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Application of a Quantum Action Principle to a Quantum Oscillator

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A quantum action principle is presented and discussed. These ideas are then applied to a specific physical example, the effect produced by an oscillator on another physical system.

1. INTRODUCTION

In physical applications, one is often interested not in individual transition probabilities, but rather in expectation values of a physical property for a specified initial state, or mixture of states. The intention here is to consider, as an extended example, one such problem, and in the process, exhibit the use of a particular quantum action principle that has been briefly introduced by Schwinger (1960, 1962, 1970).

To begin with, as an extension of Hamilton's principle in classical mechanics, suppose the action functional for a given system is $S[C]$. Then the dynamical path followed by the system in configuration space is that path about which general variations produce only end point contributions to ΔS,

$$\Delta S[C] = \Delta \int_{t_1}^{t_2} L(q_s(t), \dot{q}_s(t), t)\, dt = \left[\sum_s p_s \Delta q_s - H \Delta t \right]_{t_1}^{t_2},$$

where p_s and $-H$ in ΔS are said to be conjugate to the variables q_s and t, respectively (Sudarshan and Mukunda, 1974).

Thus, the variation ΔS in the action receives contributions only from the end points of the trajectory C, and these contributions can be expressed in terms of the total variations Δq_s, and Δt. This also leads directly to Hamilton's differential equations in the classical regime.

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The quantum version of this action principle avoids the use of functional integration because it does not directly deal with the matrix element $\langle 2 \mid 1 \rangle$ but with $\delta\langle 2 \mid 1 \rangle$, where the variations can be real or virtual. In this respect, the Schwinger principle, as it is sometimes referred to, is analogous to and based on classical variational principles, and leads to differential equations rather than to the solutions of these equations.

Schwinger (1970) defines an operator δW_{12} in the following way,

$$\delta\langle 2 \mid 1 \rangle = \frac{i}{\hbar}\langle 2|\delta W_{12}|1\rangle,$$

and for the moment, we write the quantum numbers of the states in this way. This satisfies the following additivity property

$$\delta W_{31} = \delta W_{32} + \delta W_{21}.$$

From this, it follows that the additional properties $\delta W_{11} = 0$ and $\delta W_{12} = -\delta W_{21}$ hold. Using these basic properties, it follows that

$$\frac{i}{\hbar}\langle 1|\delta W_{12}|2\rangle = \delta\langle 1 \mid 2\rangle = \delta\langle 2 \mid 1\rangle^* = -\frac{i}{\hbar}\langle 2|\delta W_{21}|1\rangle^* = \frac{i}{\hbar}\langle 2|\delta W_{12}|1\rangle^*,$$

that is

$$\langle 1|\delta W_{12}|2\rangle = \langle 2|\delta W_{21}|1\rangle^*,$$

and is Hermitian. In these terms, one can generate a dynamical postulate. There exists a special class of deformations for which the associated operators δW_{12} are obtained by appropriate variation of a single operator, the action operator.

We would now like to apply this to another elementary physical example which was not treated in Bracken (1997), but has been introduced in Schwinger (1961). In the process, it will be shown that one can write down an action principle technique that is adapted to the physical question which concerns the calculation of the expectation values of a physical property for a specified initial state and not individual transition probabilities

$$\langle X(t_2)\rangle_{b't_1} = \sum_{a'a''}\langle b't_1 \mid a't_2\rangle\langle a't_2|X|a''t_2\rangle\langle a''t_2 \mid b't_1\rangle,$$

or more generally, a mixture of states.

The action principle asserts that

$$\delta\langle a't_2 \mid b't_1\rangle = \frac{i}{\hbar}\left\langle a't_2\left|\delta\int_{t_1}^{t_2} dt\ L\right|b't_1\right\rangle,$$

in which we take $t_2 > t_1$. This particular form may be thought of as corresponding to the point of view in which states at different times can be compared by progressing forward from earlier time. The complex conjugate form would correspond to progressing from the later time

$$\delta\langle b' t_1 \mid a' t_2\rangle = -\frac{i}{\hbar}\left\langle b' t_1\middle|\delta\int_{t_1}^{t_2} dt\ L\middle|a' t_2\right\rangle.$$

Suppose that one can imagine that the two different senses of time can be thought of as being determined by different dynamics. The transformation function analogous to those above would correspond for the closed path to the action principle,

$$\delta\langle t_1 \mid t_1\rangle = \frac{i}{\hbar}\left\langle t_1\middle|\delta\left[\int_{t_1}^{t_2} dt\ L_+ - \int_{t_1}^{t_2} dt\ L_-\right]\middle|t_1\right\rangle.$$

2. FORMULATION OF THE EXAMPLE

Consider an oscillator subjected to an arbitrary external force, described by the Lagrangian operator

$$L = iy^\dagger\left(\frac{\partial y}{\partial t}\right) - \omega y^\dagger y - y^\dagger K(t) - yK^*(t). \tag{1}$$

The complementary pair of non-Hermitian operators $y,\ iy^\dagger$ are constructed from Hermitian $q,\ p$ by using the definitions,

$$y = 2^{-1/2}(q + ip), \qquad iy^\dagger = 2^{-1/2}(p + iq). \tag{2}$$

The equations of motion implied by the action principle are

$$i\left(\frac{dy}{dt}\right) - \omega y = K, \qquad -i\left(\frac{dy^\dagger}{dt}\right) - \omega y^\dagger = K^*. \tag{3}$$

Solutions are given by

$$y(t) = e^{-i\omega(t-t_2)}y(t_2) - i\int_{t_2}^{t} dt'\ e^{-i\omega(t-t')}K(t'), \tag{4}$$

as well as its adjoint equation. Differentiating (4) with respect to t, we obtain

$$\frac{dy}{dt}(t) = -i\omega\, e^{-i\omega(t-t_2)}y(t_2) - \omega\, e^{-i\omega t}\int_{t_2}^{t} dt'\, e^{i\omega t'}K(t') - ie^{i\omega t}K(t)\, e^{-i\omega t}. \tag{5}$$

The forces that are encountered in the positive time sense are written, $K_+(t)$, $K_+^*(t)$ and in the reverse time direction $K_-(t)$, $K_-^*(t)$ with $t_1 > t_2$. The integral in these cases must be taken along the appropriate path. Suppose that time t is reached first in the time evolution from t_2, then we must have from (4) that,

$$y_+(t) = e^{-i\omega(t-t_2)}y_+(t_2) - i\int_{t_2}^{t_1} dt'\, e^{-i\omega(t-t')}K_+(t'), \tag{6}$$

and on the return segment

$$y_-(t) = e^{-i\omega(t-t_2)}y_+(t_2) - i\int_{t_2}^{t_1} dt'\, e^{-i\omega(t-t')}K_+(t') + i\int_{t}^{t_1} dt'\, e^{-i\omega(t-t')}K_-(t').$$

Subtracting $y_+(t)$ from this, one obtains

$$\begin{aligned} y_-(t) - y_+(t) = -i\left(\int_{t_2}^{t_1} dt'\, e^{-i\omega(t-t')}K_+(t') - \int_{t_2}^{t} dt'\, e^{-i\omega(t-t')}K_+(t')\right) \\ + i\int_{t}^{t_1} dt'\, e^{-i\omega(t-t')}K_-(t'). \end{aligned}$$

Substituting $t = t_1$ and then $t = t_2$, one obtains that

$$\begin{aligned} &y_-(t_1) - y_+(t_1) = 0, \\ &y_-(t_2) - y_+(t_2) = i\int_{t_2}^{t_1} dt\, e^{i\omega(t-t_2)}(K_- - K_+)(t). \end{aligned} \tag{7}$$

Next, the transformation function referring to the lowest energy state of the unperturbed oscillator can be constructed. This is characterized by

$$\langle 0t_2|y^\dagger y(t_2)|0t_2\rangle = 0,$$

or equivalently, by the eigenvector equations

$$y(t_2) \mid 0t_2\rangle = 0, \qquad \langle 0t_2 \mid y^\dagger(t_2) = 0.$$

Clearly, the transformation function equals unity if $K_+ = K_-$, so there is no distinction between paths. The effect of independent changes in K_+ and K_- and of K_+^* and K_-^* must be determined using the action principle

$$\begin{aligned} &\delta_K \langle 0t_2 \mid 0t_2\rangle^{K_\pm} \\ &= -i\left\langle 0t_2\left|\left[\int_{t_2}^{t_1} dt(\delta K_+^* y_+ - \delta K_-^* y_-) + \int_{t_2}^{t_1} dt(y_+^\dagger \delta K_+ - y_-^\dagger \delta K_-)\right]\right|0t_2\right\rangle^{K_\pm}. \end{aligned}$$

The choice of initial state implies effective boundary conditions that supplement the equations of motion, thus $y_+(t_2) \to 0$, $y_-^\dagger(t_2) \to 0$. Hence,

$$y_+(t) = -i\int_{t_2}^{t_1} dt'\, e^{-i\omega(t-t')}\eta_+(t-t')K_+(t), \tag{8}$$

and

$$y_-(t) = -i\int_{t_2}^{t_1} dt'\, e^{-i\omega(t-t')}K_+(t') + i\int_{t_2}^{t_1} dt'\, e^{-i\omega(t-t')}\eta_-(t-t')K_-(t'). \quad (9)$$

There is an adjoint equation that is obtained by interchanging the $\pm$ labels. The step function $\eta_+(t-t')$ has been introduced which is 1 if $t-t' > 0$ and 0 when $t-t' < 0$, as well as $\eta_-(t-t')$ which is 1 if $t-t' < 0$ and 0 if $t-t' > 0$. Thus,

$$\eta_+(t-t') + \eta_-(t-t') = 1, \quad \eta_+(0) = \eta_-(0) = \frac{1}{2}.$$

Defining the matrices

$$K(t) = \begin{pmatrix} K_+(t) \\ K_-(t) \end{pmatrix}, \quad iG_0(t-t') = e^{-i\omega(t-t')}\begin{pmatrix} \eta_+(t-t') & 0 \\ -1 & \eta_-(t-t') \end{pmatrix}, \quad (10)$$

one can calculate that,

$$\begin{aligned} &iK^*(t)G_0(t-t')K(t') \\ &\quad = i(K_+^*(t)K_-^*(t))\begin{pmatrix} \eta_+(t-t') & 0 \\ -1 & \eta_-(t-t') \end{pmatrix}\begin{pmatrix} K_+(t') \\ K_-(t') \end{pmatrix} e^{-i\omega(t-t')} \\ &\quad = i\, e^{-i\omega(t-t')}\begin{pmatrix} K_+^*(t)\eta_+(t-t')K_+(t) \\ -K_-^*(t)K_+(t') + K_-^*(t)\eta_-(t-t')K_-(t') \end{pmatrix}. \end{aligned}$$

One can then write the solution of the integrable differential expression for $\langle 0t_2 \mid 0t_2\rangle^{K_\pm}$ in matrix form

$$\langle 0t_2 \mid 0t_2\rangle^{K_\pm} = \exp\left[-i\int_{t_2}^{t_1} dt\, dt'\, K^*(t)G_0(t-t')K(t')\right]. \quad (11)$$

The second variation of this gives

$$-\delta_{K^*}\delta_K\langle 0t_2 \mid 0t_2\rangle_{K_\pm}|_{K=K^*=0} = i\int_{t_2}^{t_1} dt\, dt'\, \delta K^*(t)G_0(t-t')\,\delta K(t'). \quad (12)$$

The results for any initial oscillator state can be derived. To do this, consider the impulsive forces,

$$K_+(t) = iy''\delta(t-t_2), \qquad K_-^*(t) = -iy^{\dagger'}\delta(t-t_2). \quad (13)$$

Thus, under the influence of these forces, the states $|0t_2\rangle$ and $\langle 0t_2|$ become at t_2+0, the states $|y''t_2\rangle$ and $\langle y^{\dagger'}t_2|$, which are right and left eigenvectors of $y(t_2)$ and $y^\dagger(t_2)$. The transformation function for the closed time path, on taking into account arbitrary additional forces, can be expressed as

$$\langle y^{\dagger'} t_1 \mid y'' t_2 \rangle^{K_\pm} = \exp\Bigg[y^{\dagger'} y'' - y^{\dagger'} \left(-i \int_{t_2}^{t_1} dt\; e^{i\omega(t-t_2)} \,(K_- - K_+)\,(t) \right) + \left(-i \int_{t_2}^{t_1} dt\; e^{-i\omega(t-t_2)} \,(K_+^* - K_-^*)\,(t) \right) y'' - i \int_{t_2}^{t_1} dt\; dt'\; K^*(t) G_0(t-t') K(t') \Bigg]. \tag{14}$$

Unperturbed oscillator energy states will be of interest here. These two descriptions can be connected by considering the unperturbed oscillator transformation function

$$\langle y^{\dagger'} t_1 \mid y'' t_2 \rangle = \langle y^{\dagger'} | \exp[-i(t_1 - t_2)\omega y^\dagger y] | y'' \rangle. \tag{15}$$

Differentiating this with respect to t_1, one obtains that

$$i \frac{\partial}{\partial t_1} \langle y^{\dagger'} t_1 \mid y'' t_2 \rangle = \langle y^{\dagger'} t_1 | \omega y^\dagger(t_1) y(t_1) | y'' t_2 \rangle = \omega y^{\dagger'}\; e^{-i\omega(t_1-t_2)} y'' \langle y^{\dagger'} t_1 \mid y'' t_2 \rangle, \tag{16}$$

since,

$$y(t_1) = e^{-i\omega(t_1-t_2)} y(t_2).$$

Integrating (16), we obtain the explicit expression,

$$\langle y^{\dagger'} t_1 \mid y'' t_2 \rangle = \exp\left[y^{\dagger'}\; e^{-i\omega(t_1-t_2)} y'' \right]. \tag{17}$$

The exponential can be expanded to give,

$$\langle y^{\dagger'} t_1 \mid y'' t_2 \rangle = \exp\left[y^{\dagger'} e^{-i\omega(t_1-t_2)} y'' \right] = \sum_{n=0}^{\infty} \frac{(y^{\dagger'})^n}{\sqrt{n!}}\; e^{-in\omega(t_1-t_2)} \frac{(y'')^n}{\sqrt{n!}}. \tag{18}$$

Suppose we are interested in the expectation values that refer to initial state n, namely $\langle n t_2 \mid n t_2 \rangle^{K_\pm}$. The coefficient of $(y^{\dagger'} y'')^n / n!$ must be extracted from an exponential of the form

$$\exp[y^{\dagger'} y'' + y^{\dagger'} \alpha + \beta y'' + \gamma] = \sum_{kl} \frac{(y^{\dagger'})^k}{k!} \frac{(y'')^l}{l!} \alpha^k \beta^l \exp[y^{\dagger'} y'' + \gamma]. \tag{19}$$

All the terms that contribute to the matrix element are contained in the diagonal part of the sum

$$\sum_{k=0}^{\infty} \frac{(y^{\dagger'} y'')^k}{(k!)^2} (\alpha\beta)^k\; \exp[y^{\dagger'} y'' + \gamma] = \frac{1}{2\pi i} \int \frac{ds}{s}\; e^s \exp[y^{\dagger'} y'' (1 + s^{-1} \alpha\beta) + \gamma], \tag{20}$$

and one takes a path so that the integral is given by

$$\frac{1}{2\pi i}\int \frac{ds}{s^{k+1}}\, e^{s} = \frac{1}{k!}.$$

The calculation proceeds as follows

$$\begin{aligned}\frac{1}{2\pi i}\int \frac{ds}{s}\, e^{s}\ \exp[y^{\dagger'} y''(s\alpha\beta)] &= \frac{1}{2\pi i}\int \sum_{k=0}^{\infty} \frac{ds}{s^{k+1}}\, e^{s} \frac{(y^{\dagger'} y'')^{k}}{k!}(\alpha\beta)^{k}\\ &= \sum_{k=0}^{\infty} \frac{(y^{\dagger'} y'')^{k}}{k!^{2}}(\alpha\beta)^{k}\\ &= \sum_{k=0}^{\infty} \frac{(y^{\dagger'})^{k}}{\sqrt{k!}}\frac{(y'')^{k}}{\sqrt{k!}}\frac{(\alpha\beta)^{k}}{k!}.\end{aligned}$$

The quantities $\alpha,\ \beta$, and γ are obtained by compairing with the expression for $\langle y^{\dagger'} t_1 \mid y'' t_2\rangle^{K_\pm}$,

$$\begin{aligned}&\frac{1}{2\pi i}\int \frac{ds}{s}\, e^{s} \exp[y^{\dagger'} y''(1+s^{-1}\alpha\beta)] \exp(\gamma)\\ &\quad = \frac{1}{2\pi i}\int \frac{ds}{s}\, e^{s} \sum_{n} \frac{(y^{\dagger'} y'')^{n}}{n!}(1+s^{-1}\alpha\beta)^{n} = \sum_{n} \frac{(y^{\dagger'} y'')}{n!} L_{n}(-\alpha\beta).\end{aligned}$$

One obtains a much nicer form, however, by considering an initial mixture of oscillator energy states for which the nth state is assigned the probability

$$(1-e^{-\beta\omega})\, e^{-n\beta\omega}. \tag{21}$$

These results are recovered in this way, and $\beta = \theta^{-1}$ can be interpreted as a temperature. Since,

$$\begin{aligned}&(1-e^{-\beta\omega})\sum_{n=0}^{\infty} e^{-n\beta\omega} L_{n}(x)\\ &\quad = (1-e^{-\beta\omega})\frac{1}{2\pi i}\int \frac{ds}{s}\, e^{s}(1-e^{-\beta\omega}+s^{-1}e^{-\beta\omega}x)^{-1} = \exp\left(-\frac{x}{e^{\beta\omega}-1}\right).\end{aligned}$$

One obtains, suppressing θ on the matrix element,

$$\langle t_2 \mid t_2\rangle_{\theta}^{K_\pm} = \langle t_2 \mid t_2\rangle^{K_\pm} = \exp\left[-i\int dt\, dt'\, K^{*}(t) G_{\theta}(t-t') K(t')\right], \tag{22}$$

where,

$$iG_{\theta}(t-t') = iG_{0}(t-t') + (e^{\beta\omega}-1)^{-1} G_{0}(t-t_2)_{+-} G_{0}(t_2-t'),$$

$$iG_{0}(t-t_2)_{+} = e^{-i\omega(t-t_2)}\begin{pmatrix}1\\-1\end{pmatrix}, \qquad i_{-}G_{0}(t_2-t) = e^{i\omega(t-t_2)}(-1\quad 1).$$

Thus,

$$iG_\theta(t-t') = e^{-i\omega(t-t')}\begin{pmatrix} \eta_+(t-t') + \langle n\rangle_\theta & -\langle n\rangle_\theta \\ -1-\langle n\rangle_\theta & \eta_-(t-t') + \langle n\rangle_\theta \end{pmatrix}, \tag{23}$$

where we have denoted,

$$\langle n\rangle_\theta = \frac{1}{e^{\beta\omega}-1}. \tag{24}$$

Since the elements of G_θ are also given by unperturbed oscillator thermal expectation values

$$iG_\theta(t-t') = \begin{pmatrix} \langle (y(t)y^\dagger(t'))_+\rangle_\theta & -\langle y^\dagger(t')y(t)\rangle_\theta \\ -\langle y(t)y^\dagger(t')\rangle_\theta & \langle (y(t)y^\dagger(t'))_-\rangle_\theta \end{pmatrix}, \tag{25}$$

and the notation $\langle n\rangle_\theta$ is consistent with its identification as $\langle y^\dagger y\rangle_\theta$.

The thermal forms can also be derived directly by solving the equations of motion, in the manner used to find $\langle 0t_2 \mid 0t_2\rangle^{K_\pm}$. On replacing the single diagonal element

$$\langle 0t_2 \mid 0t_2\rangle^{K_\pm} = \langle 0t_2|U|0t_2\rangle,$$

by the statistical average,

$$(1-e^{-\beta\omega})\sum_{n=0}^{\infty} e^{-n\beta\omega}\langle nt_2 \mid nt_2\rangle^{K_\pm} = (1-e^{-\beta\omega})\,\mathrm{tr}[\exp(-\beta\omega y^\dagger y)U], \tag{26}$$

we find the following relation,

$$y_-(t_2) = e^{\beta\omega}y_+(t_2), \tag{27}$$

instead of the effective initial condition $y_+(t_2) = 0$. This is obtained by combining

$$\exp(-\beta\omega y^\dagger y)y\ \exp(\beta\omega y^\dagger y) = \exp(\beta\omega)y, \tag{28}$$

with the property of the trace,

$$\begin{aligned}\mathrm{tr}(\exp(-\beta\omega y^\dagger y)yU) &= \mathrm{tr}(\exp(\beta\omega)y\ \exp(-\beta\omega y^\dagger y)U) \\ &= \mathrm{tr}(\exp(-\beta\omega y^\dagger y)U\ \exp(\beta\omega)y).\end{aligned}$$

We also have,

$$y_-(t_2) - y_+(t_2) = -i\int_{t_2}^{t_1} dt\ e^{i\omega(t-t_2)}(K_+ - K_-)(t). \tag{29}$$

Using (27) in (29), we obtain that,

$$y_+(t_2) = -\frac{i}{e^{\beta\omega} - 1} \int_{t_2}^{t_1} dt\, e^{i\omega(t-t_2)}(K_+ - K_-)(t).$$

Hence, to the previously determined $y_\pm(t)$ is to be added the term

$$-i\langle n\rangle_\theta \int_{t_2}^{t_1} dt'\, e^{-i\omega(t-t')}(K_+ - K_-)(t),$$

and correspondingly,

$$\langle t_2 \mid t_2\rangle_\theta^{K_\pm} = \langle t_2 \mid t_2\rangle_0^{K_\pm} \exp\left[-\langle n\rangle_\theta \int_{t_2}^{t_1} dt\, dt'(K_+^* - K_-^*)(t) \right.$$
$$\left. \times\, e^{-i\omega(t-t')}(K_+ - K_-)(t') \right]. \tag{30}$$

As an easier example, let us evaluate the expectation value of the oscillator energy at time t_1 for a system that was in thermal equilibrium at time t_2, and is subsequently disturbed by an arbitrary time-varying force. This can be computed as follows,

$$\langle t_2|\omega y^\dagger y(t_1)|t_2\rangle_\theta^K = \omega \frac{\delta}{\delta K_-(t_1)} \frac{\delta}{\delta K_+^*(t_1)} \langle t_2 \mid t_2\rangle_\theta^{K_\pm}\Big|_{K_+=K_-, K_+^*=K_-^*}. \tag{31}$$

The variation with respect to $\delta/\delta K_+^*(t_1)$ gives the factor

$$-i\left(\int_{t_2}^{t_1} dt\, G_\theta(t_1 - t')K(t') \right)_+,$$

The subsequent variation with respect to $K_-(t_-)$ gives

$$-iG_\theta(0)_{+-} + \left(\int dt\, K^*(t)G_\theta(t - t_1) \right)_- \left(\int dt'\, G_\theta(t_1 - t')K(t') \right)_+. \tag{32}$$

The required energy expectation value equals,

$$\omega\langle n\rangle_\theta + \omega\left| \int_{t_2}^{t_1} dt\, e^{i\omega t} K(t) \right|^2. \tag{33}$$

More generally, the expectation values of all functions of $y(t_1)$ and $y^\dagger(t_1)$ are known by finding those of

$$\exp[-i(\lambda y^\dagger(t_1) + \mu y(t_1))],$$

and this quantity is obtained on supplementing K_+ and K_+^* by the impulsive forces,

$$K_+(t) = \lambda\delta(t - t_1), \qquad K_+^*(t) = \mu\delta(t - t_1), \tag{34}$$

then,

$$\langle t_2|\exp[-i(\lambda y^\dagger(t_1) + \mu y(t_1))]|t_2\rangle_\theta^K$$
$$= \exp\left[-\lambda\mu\left(\langle n\rangle_\theta + \frac{1}{2}\right) + \lambda\int_{t_2}^{t_1} dt\, e^{i\omega(t_1-t)}K^*(t)\right.$$
$$\left.-\mu\int_{t_2}^{t_1} dt\, e^{-i\omega(t_1-t)}K(t)\right], \tag{35}$$

where we use $\eta_+(0) = 1/2$.

If probabilities for specific oscillator energy states are of interest, we have only to exhibit as functions of y and $y^\dagger$, the projection operators for these states, the expectation values of which are the required probabilities. The operator

$$P_n = |n\rangle\langle n|,$$

is represented by the matrix

$$\langle y^{\dagger'}|P_n|y''\rangle = \frac{(y^{\dagger'}y'')^{2n}}{n!}\ \exp(-y^{\dagger'}y'')\langle y^{\dagger'}\ |\ y''\rangle, \tag{36}$$

and therefore,

$$P_n = \frac{1}{n!}(y^\dagger)^n\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(y^\dagger)^k y^k\right]y^n = \frac{1}{n!}(y^\dagger)^n\ \exp(-y^\dagger; y)y^n,$$

in which we have introduced a notation to indicate the ordered multiplication of operators. A convenient generating function for these projection operators is

$$\sum_{n=0}^{\infty}\alpha^n P_n = \exp[-(1-\alpha)y^\dagger; y].$$

This can be written,

$$\sum_{n=0}^{\infty}\alpha^n P_n = \exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\exp(-i\lambda y^\dagger)\exp(-i\mu y)|_{\lambda=\mu=0}.$$

Then,

$$\sum_{n=0}^{\infty}\alpha^n p(n,\theta,K)$$
$$= \exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\ \exp\left[-\lambda\mu\langle n\rangle_\theta + \lambda\, e^{i\omega t_1}\int dt\, e^{i\omega t}K^*(t)\right.$$
$$\left.-\mu\, e^{-i\omega t_1}\int dt\, e^{i\omega t}K(t)\right]_{\lambda=\mu=0}, \tag{37}$$

gives the probability of finding the oscillator in the nth energy state after an arbitrary time-varying force has acted, if it was initially in a thermal mixture of states.

To evaluate the quantity,

$$X = \exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\exp[-\lambda\mu\langle n\rangle + \lambda\gamma^* - \mu\gamma]\,|_{\lambda=\mu=0},$$

differentiate X with respect to γ^*, and with $\langle n\rangle = \langle n\rangle_\theta$, we have,

$$\begin{aligned}\frac{\partial}{\partial\gamma^*}X &= \exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\lambda\ \exp[-\lambda\mu\langle n\rangle + \lambda\gamma^* - \mu\gamma]\,|_{\lambda=\mu=0}\\ &= (1-\alpha)\ \exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\frac{\partial}{\partial\mu}\ \exp[-\lambda\mu\langle n\rangle + \lambda\gamma^* - \mu\gamma]\,|_{\lambda=\mu=0}\\ &= (1-\alpha)\ \exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right](-\lambda\langle n\rangle - \gamma)\\ &\quad\times \exp[-\lambda\mu\langle n\rangle + \lambda\gamma^* - \mu\gamma]\,|_{\lambda=\mu=0}.\end{aligned}$$

Using the Cambell–Baker–Hausdorf formula, one has that,

$$\begin{aligned}&\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right](-\lambda\langle n\rangle - \gamma)\ \exp\left[-(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\\ &\quad= (-\lambda\langle n\rangle - \gamma) - (1-\alpha)\langle n\rangle\frac{\partial}{\partial\mu}.\end{aligned}$$

Since,

$$\begin{aligned}&(1-\alpha)\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\frac{\partial}{\partial\mu}\ \exp[-\lambda\mu\langle n\rangle + \lambda\gamma^* - \mu\gamma]\,|_{\lambda=\mu=0}\\ &\quad= \exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\lambda\ \exp[-\lambda\mu\langle n\rangle + \lambda\gamma^* - \mu\gamma]\,|_{\lambda=\mu=0} = \frac{\partial X}{\partial\gamma^*},\end{aligned}$$

one obtains the following differential equation in X,

$$\frac{\partial X}{\partial\gamma^*} = -(1-\alpha)\gamma X - (1-\alpha)\langle n\rangle\frac{\partial X}{\partial\gamma^*}.$$

This can be rewritten in the form

$$\frac{\partial X}{\partial\gamma^*} = -\frac{\gamma(1-\alpha)}{1+\langle n\rangle(1-\alpha)}X. \tag{38}$$

Solving this equation, one can write

$$X = X_0\ \exp\left[-|\gamma|^2\frac{1-\alpha}{1-\langle n\rangle(1-\alpha)}\right],$$

where

$$X_0 = \frac{1}{1 + \langle n \rangle (1 - \alpha)}.$$

Therefore, substituting $\langle n \rangle = (e^{\beta\omega} - 1)^{-1}$ into X_0, we can write,

$$X_0 = [1 + (e^{\beta\omega} - 1)^{-1}(1 - \alpha)]^{-1} = (1 - e^{-\beta\omega})\,(1 - \alpha\, e^{-\omega\beta})^{-1}.$$

Thus the sum (37) is given by

$$\sum_{n=0}^{\infty} \alpha^n p(n, \theta, K) = \frac{1 - e^{-\beta\omega}}{1 - \alpha\, e^{-\omega\beta}}\ \exp\left[-|\gamma|^2 \frac{1 - e^{-\beta\omega}}{1 - \alpha\, e^{-\beta\omega}} (1 - \alpha) \right],$$

where $|\gamma|^2 = |\int dt\, e^{i\omega t} K(t)|^2$. On refering to the previously used Laguerre polynomial sum formula, we obtain,

$$p(n, \theta, K) = (1 - e^{-\beta\omega})\, e^{-n\beta\omega} \exp[-|\gamma|^2(1 - e^{-\beta\omega})]\, L_n[-4|\gamma|^2 \sinh^2(\beta\omega/2)].$$

In addition to describing the physical situation of initial thermal equilibrium, this also provides a generating function for the individual transition probabilities between oscillator energy states,

$$\sum_{n'=0}^{\infty} p(n, n', K)\, e^{-(n'-n)\beta\omega}$$
$$= \exp[-|\gamma|^2(1 - e^{-\beta\omega})]\, L_n[-(1 - e^{-\beta\omega})\,(e^{\beta\omega} - 1)|\gamma|^2].$$

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